K-theory for the crossed products of infinite tensor product of C^*-algebras by the shift

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Abstract

C. Schochet shows Künneth theorem for the C^*-algebras in the smallest class of nuclear C^*-algebras which contains the separable Type I algebras and is closed under some operations. We calculate the K-theory for the crossed product of the infinite tensor product of a unital C^*-algebra in this class by the shift. In particular, we calculate the K-theory of the lamplighter group C^*-algebra.

1 Introduction

Let A be a unital C^*-algebra and $A^{\otimes n}$ denote the (minimal) tensor product of n copies of A for each $n \geq 1$. An *infinite tensor product* $A^{\otimes \mathbb{Z}}$ of A is the inductive limit of the sequence $\{A^{\otimes 2n-1}\}_n$ with the embeddings $a \mapsto 1_A \otimes a \otimes 1_A$. Note that $A^{\otimes \mathbb{Z}}$ is represented as the closed linear span of

$$\left\{ \bigotimes_{i \in \mathbb{Z}} a_i \ : \ a_i \in A,\ a_j = 1_A \text{ for all but finitely many } j \right\}.$$

A *shift* s on $A^{\otimes \mathbb{Z}}$ is the automorphism of $A^{\otimes \mathbb{Z}}$ satisfying

$$s\left(\bigotimes_{i \in \mathbb{Z}} a_i\right) = \bigotimes_{i \in \mathbb{Z}} a_{i-1} \qquad \text{for all } \bigotimes_{i \in \mathbb{Z}} a_i \in A^{\otimes \mathbb{Z}}.$$

We consider the crossed product $A^{\otimes \mathbb{Z}} \rtimes_s \mathbb{Z}$ of $A^{\otimes \mathbb{Z}}$ by $\mathbb{Z}$ under the action by s.

Example 1. (i) Let $C(X)$ denote the C^*-algebra of all continuous functions on a compact Hausdorff space X and θ be an automorphism on $X^{\mathbb{Z}}$ defined by $(x_i)_{i \in \mathbb{Z}} \mapsto (x_{i+1})_{i \in \mathbb{Z}}$. Then $C(X^{\mathbb{Z}}) \rtimes_\theta \mathbb{Z} \cong C(X)^{\otimes \mathbb{Z}} \rtimes_s \mathbb{Z}$.

(ii) Let $\Gamma \wr \mathbb{Z}$ denote the *wreath product* of an amenable discrete group Γ with $\mathbb{Z}$. Then, since the group C^*-algebra $C^*(\Gamma)$ of Γ is nuclear, $C^*(\Gamma \wr \mathbb{Z}) \cong C^*(\Gamma)^{\otimes \mathbb{Z}} \rtimes_s \mathbb{Z}$.

For a C^*-algebra B, $K_*(B)$ is the ($\mathbb{Z}_2$-)graded K-theory $K_0(B) \oplus K_1(B)$, where $\mathbb{Z}_2 := \mathbb{Z}/2\,\mathbb{Z}$. Let $\mathcal{N}$ denote the smallest class of separable nuclear C^*-algebras which contains the separable Type I algebras and is closed under the operations of taking ideals, quotients, extensions, inductive limits, stable isomorphism, and crossed products by $\mathbb{Z}$ or $\mathbb{R}$. The class $\mathcal{N}$ plays an essential role in Künneth theorem by C. Schochet:

Theorem 1 (the Spatial Case of Künneth Theorem [S, Theorem 2.14])**.** *Let A and B be C^*-algebras such that A is contained in $\mathcal{N}$. If $K_*(B)$ is torsion free, then there exists a natural isomorphism*

$$\alpha : K_*(A) \otimes K_*(B) \longrightarrow K_*(A \otimes B),$$

that is, there exist a natural isomorphisms

$$\begin{aligned}(K_0(A) \otimes K_0(B)) \oplus (K_1(A) \otimes K_1(B)) &\longrightarrow K_0(A \otimes B),\\ (K_0(A) \otimes K_1(B)) \oplus (K_1(A) \otimes K_0(B)) &\longrightarrow K_1(A \otimes B).\end{aligned}$$

We introduce the concept of infinite tensor product of graded abelian group. Let $G = G_0 \oplus G_1$ be an graded abelian group and $G^{\otimes n}$ denote the tensor product of n copies of G (as graded $\mathbb{Z}$-module). For an elements e of G_0, an *infinite tensor product* $(G,e)^{\otimes\,\mathbb{Z}}$ of G with respect to e is the inductive limit of the sequence $\{G^{\otimes 2n-1}\}_n$ with the embeddings $x \mapsto e \otimes x \otimes e$. We often denote $(G,e)^{\otimes\,\mathbb{Z}}$ by $G^{\otimes\,\mathbb{Z}}$. Note that $(G,e)^{\otimes\,\mathbb{Z}}$ is represented as the linear span of

$$\left\{ \bigotimes_{i\in\mathbb{Z}} x_i \ : \ x_i \in G,\ x_j = e \text{ for all but finitely many } j \right\}$$

with the grading $(G^{\otimes\,\mathbb{Z}})_0 \oplus (G^{\otimes\,\mathbb{Z}})_1$, where $(G^{\otimes\,\mathbb{Z}})_p$ is the linear span of

$$\left\{ \bigotimes_{i\in\mathbb{Z}} x_i \in (G,e)^{\otimes\,\mathbb{Z}} \ : \ x_i \in G_{p_i},\ p_i \in \mathbb{Z}_2,\ \sum_{i\in\mathbb{Z}} p_i = p \right\}.$$

A *shift* λ on $(G,e)^{\otimes\,\mathbb{Z}}$ is the automorphism of $(G,e)^{\otimes\,\mathbb{Z}}$ satisfying

$$\lambda\left(\bigotimes_{i\in\mathbb{Z}} x_i\right) = \bigotimes_{i\in\mathbb{Z}} x_{i-1} \qquad \text{for all } \bigotimes_{i\in\mathbb{Z}} x_i \in (G,e)^{\otimes\,\mathbb{Z}}.$$

Remark 1. Let A be a unital C^*-algebra in $\mathcal{N}$, and assume that $K_*(A)$ is a graded *free* abelian group.

(i) Künneth theorem gives that there exists a natural isomorphism α_n from $K_*(A)^{\otimes n}$ to $K_*(A^{\otimes n})$ for each $n \geq 1$. Form the definition of α [S, p.446], there exists a natural isomorphism α_∞ from $(K_*(A), [1_A]_0)^{\otimes\,\mathbb{Z}}$ to $K_*(A^{\otimes\,\mathbb{Z}})$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc} K_*(A) & \longrightarrow & K_*(A)^{\otimes 3} & \longrightarrow & K_*(A)^{\otimes 5} & \longrightarrow & \cdots & \longrightarrow & K_*(A)^{\otimes\,\mathbb{Z}} \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_3} & & \downarrow{\scriptstyle \alpha_5} & & & & \downarrow{\scriptstyle \alpha_\infty} \\ K_*(A) & \longrightarrow & K_*(A^{\otimes 3}) & \longrightarrow & K_*(A^{\otimes 5}) & \longrightarrow & \cdots & \longrightarrow & K_*(A^{\otimes\,\mathbb{Z}}). \end{array} \tag{1}$$

(ii) The automorphism $\lambda := \alpha_\infty^{-1} \circ s_* \circ \alpha_\infty$ is the shift on $(K_*(A), [1_A]_0)^{\otimes \mathbb{Z}}$.

(iii) Applying $A^{\otimes \mathbb{Z}} \rtimes_s \mathbb{Z}$ to Pimsner–Voiculescu theorem [B, Theorem 10.2.1], we obtain the following exact triangle:

$$\begin{tikzcd} K_*(A^{\otimes \mathbb{Z}}) \arrow[rr, "\mathbf{id} - s_*"] & & K_*(A^{\otimes \mathbb{Z}}) \arrow[dl, "\iota_*"] \\ & K_*(A^{\otimes \mathbb{Z}} \rtimes_s \mathbb{Z}), \arrow[ul, "\partial"] & \end{tikzcd}$$

where $\mathbf{id} - s_*$ and ι_* have degree 0 and ∂ has degree 1. Therefore, by (i) and (ii), we obtain the following exact triangle:

$$\begin{tikzcd} K_*(A)^{\otimes \mathbb{Z}} \arrow[rr, "\mathbf{id} - \lambda"] & & K_*(A)^{\otimes \mathbb{Z}} \arrow[dl, "\iota_* \circ \alpha_\infty"] \\ & K_*(A^{\otimes \mathbb{Z}} \rtimes_s \mathbb{Z}). \arrow[ul, "\alpha_\infty^{-1} \circ \partial"] & \end{tikzcd} \tag{2}$$

Let G be a free abelian group which has a basis E containing e. Let $\check{G}$ be the subgroup of G such that

$$G = \mathbb{Z}\, e \oplus \check{G}. \tag{3}$$

Let $\check{G} \otimes G^{\otimes \mathbb{N}}$ denote the linear span of

$$\left\{ \bigotimes_{i \in \mathbb{Z}} x_i \in G^{\otimes \mathbb{Z}} \ : \ x_0 \in \check{G},\ x_i = e \text{ for all } i < 0 \right\}.$$

Set $H(G, e) := \mathbb{Z}\, e^{\otimes \mathbb{Z}} + \check{G} \otimes G^{\otimes \mathbb{N}}$ where $e^{\otimes \mathbb{Z}} := \bigotimes_{i \in \mathbb{Z}} e$. Then, we prove in Lemma 3 that

$$G^{\otimes \mathbb{Z}} = (\mathbf{id} - \lambda)(G^{\otimes \mathbb{Z}}) \oplus H(G, e),$$

where λ is the shift on $G^{\otimes \mathbb{Z}}$, In particular, $H(G, e)$ is naturally isomorphic to the cokernel of $\mathbf{id} - \lambda$.

Remark 2. (i) If G is an *ordered* free abelian group with the positive cone G_+ and $e \in G_+$, then $(G, e)^{\otimes \mathbb{Z}}$ is also an ordered abelian group with the positive cone $G_+^{\otimes \mathbb{Z}}$, all sums of the elements of

$$\left\{ \bigotimes_{i \in \mathbb{Z}} x_i \in (G, e)^{\otimes \mathbb{Z}} \ : \ x_i \in G_+ \right\}.$$

Set $H(G, e)_+ := H(G, e) \cap G_+^{\otimes \mathbb{Z}}$.

(ii) Let $E^{\otimes \mathbb{Z}}$ denote the basis $\left\{ \bigotimes_{i \in \mathbb{Z}} e^{(i)} \ : \ e^{(i)} \in E,\ e^{(j)} = e \text{ for all but finitely many } j \right\}$ of $(G, e)^{\otimes \mathbb{Z}}$. Then, there exists a natural isomorphism

$$\Phi(G, e, E) \ : \ (G, e)^{\otimes \mathbb{Z}} \longrightarrow \mathbb{Z}^{\oplus (E^{\otimes \mathbb{Z}})}.$$

But, $\Phi(G, e, E)$ may *not* be order-preserving, where the positive cone of $\mathbb{Z}^{\oplus I}$ is $\mathbb{Z}_+^{\oplus I} := \{(k_m)_{m \in I} \in \mathbb{Z}^{\oplus I} \ : \ k_m \geq 0\}$ for a set I.

Example 2. (i) $(\mathbb{Z},1)^{\otimes \mathbb{Z}} = H(\mathbb{Z},1) = \mathbb{Z}\,1^{\otimes \mathbb{Z}}$.

(ii) For a natural number $d \geq 2$, let E_d be the standard basis $\{e^{(k)}\}_{k=1}^{d}$ of $\mathbb{Z}^d$. Then, $\Phi(\mathbb{Z}^d, e^{(1)}, E_d)$ is order-preserving. On the other hand, let $\tilde{E}_d$ be the basis $\{e^{(0)}\} \cap \{e^{(k)}\}_{k=2}^{d}$ where $e^{(0)} := (1,1,\cdots,1) \in \mathbb{Z}^d$. Then, $\Phi(\mathbb{Z}^d, e^{(0)}, \tilde{E}_d)$ is *not* order-preserving.

The following is our main theorem:

Main Theorem. *Let A be a C^*-algebra in $\mathcal{N}$ and assume that $K_*(A)$ is a free abelian group which has a basis containing $[1_A]_0$. Then there exists the following split exact sequence:*

$$0 \longrightarrow H(K_*(A),[1_A]_0) \xrightarrow{\varphi} K_*(A^{\otimes \mathbb{Z}} \rtimes_s \mathbb{Z}) \xrightarrow{\psi} \mathbb{Z}\,[1_A]_0^{\otimes \mathbb{Z}} \longrightarrow 0, \tag{4}$$

where φ has degree 0 and ψ has degree 1. In particular, φ is the restriction of $\iota_* \circ \alpha_\infty$ and ψ is the homomorphism defined by $x \mapsto (\alpha_\infty^{-1} \circ \partial)(x)$.

Corollary. *Let A be a C^*-algebra satisfying the assumption of the above theorem. If $K_1(A) = 0$, then*

$$H(K_0(A),[1_A]_0) \xrightarrow{\varphi} K_0(A^{\otimes \mathbb{Z}} \rtimes_s \mathbb{Z}), \qquad K_1(A^{\otimes \mathbb{Z}} \rtimes_s \mathbb{Z}) \xrightarrow{\psi} \mathbb{Z}\,[1_A]_0^{\otimes \mathbb{Z}}$$

are isomorphisms. Moreover, if $(K_0(A), K_0(A)_+)$ is ordered abelian group, then φ is order-preserving.

Example 3. We calculate the K-theory for the group C^*-algebra of the *lamplighter group*:

$$\mathbb{Z}_2 \wr \mathbb{Z} := \mathbb{Z}_2^{\oplus \mathbb{Z}} \rtimes \mathbb{Z}\,.$$

Note that

(i) $C^*(\mathbb{Z}_2 \wr \mathbb{Z}) \cong (\mathbb{C}^2)^{\otimes \mathbb{Z}} \rtimes_s \mathbb{Z}$ (Example 1),

(ii) $\mathbb{C}^2 \in \mathcal{N}$,

(iii) $(K_0(\mathbb{C}^2), K_0(\mathbb{C}^2)_+, [1_{\mathbb{C}^2}]_0) \cong (\mathbb{Z}^2, \mathbb{Z}^2_+, (1,1))$ and $K_1(\mathbb{C}^2) = 0$.

Applying the Main Theorem in the case of $A = \mathbb{C}^2$, we obtain the following isomorphisms:

$$(K_0(C^*(\mathbb{Z}_2 \wr \mathbb{Z})), K_0(C^*(\mathbb{Z}_2 \wr \mathbb{Z}))_+) \cong (H(\mathbb{Z}^2,(1,1)), H(\mathbb{Z}^2,(1,1))_+), \quad K_1(C^*(\mathbb{Z}_2 \wr \mathbb{Z})) \cong \mathbb{Z}\,.$$

2 Proof

In this section, let G be a graded free abelian group which has a basis E containing e. Set $e^{\otimes n} := \bigotimes_{j=1}^{n} e \in G^{\otimes n}$ and $G^{\otimes 0} := \mathbb{Z}$. Notice that, by (3),

$$G^{\otimes n} = (e \otimes G^{\otimes n-1}) \oplus (\check{G} \otimes G^{\otimes n-1}) \tag{5}$$

for each $n \geq 1$.

Lemma 1. *For $n \geq 1$, $y \in G^{\otimes n}$, $k \in \mathbb{Z}$, and $x \in \check{G} \otimes G^{\otimes n}$, if the equality*

$$y \otimes e - e \otimes y = ke^{\otimes n+1} + x \tag{6}$$

holds, then $y \in \mathbb{Z}\, e^{\otimes n}$, $k = 0$, and $x = 0$.

Proof. Note that, by (5), $y = e \otimes y_1 + x_1$ for some $y_1 \in G^{\otimes n-1}$ and $x_1 \in \check{G} \otimes G^{\otimes n-1}$. Then, from (6), we get the following:

$$e \otimes (y_1 \otimes e - y - ke^{\otimes n}) + (x_1 \otimes e - x) = 0.$$

Hence, by (5), $y_1 \otimes e - y - ke^{\otimes n} = 0$ and $x_1 \otimes e - x = 0$, so

$$y = y_1 \otimes e - ke^{\otimes n}, \qquad x = x_1 \otimes e.$$

Therefore, from (6), we get $(y_1 \otimes e - e \otimes y_1) \otimes e = (ke^{\otimes n} + x_1) \otimes e$, so

$$y_1 \otimes e - e \otimes y_1 = ke^{\otimes n} + x_1.$$

Note that, replacing $(n-1, y_1, x_1)$ of this equation with (n, y, x), we get (6).

Repeating this argument, we can find $y_j \in G^{\otimes n-j}$ for each integer j $(1 \leq j \leq n)$ such that

$$y_{j-1} = y_j \otimes e - ke^{\otimes n-j+1},$$

where $y_0 := y$. Inductively, we obtain $y = (y_n - nk)e^{\otimes n} \in \mathbb{Z}\, e^{\otimes n}$. Hence, from (6), $e \otimes (ke^{\otimes n}) + x = 0$. Thus, by (5), $k = 0$ and $x = 0$. $\square$

For each $n \geq 1$, identify $G^{\otimes 2n-1}$ and the linear span of

$$\left\{ \bigotimes_{i \in \mathbb{Z}} x_i \in G^{\otimes \mathbb{Z}} \ : \ x_i = e \text{ for all } |i| > 2n-1 \right\}.$$

Lemma 2. *Let λ be the shift on $G^{\otimes \mathbb{Z}}$. For $z \in G^{\otimes \mathbb{Z}}$ and $w \in H(G, e)$, if the equality*

$$(\mathbf{id} - \lambda)(z) = w \tag{7}$$

holds, then $z \in \mathbb{Z}\, e^{\otimes \mathbb{Z}}$ and $w = 0$.

Proof. Take $n \geq 1$, satisfying $(\mathbf{id} - \lambda)(z)$ $(= w) \in G^{\otimes 2n+1}$. Note that

$$(\mathbf{id} - \lambda)(G^{\otimes \mathbb{Z}}) \cap G^{\otimes 2n+1} = \{ e \otimes z' - z' \otimes e \ : \ z' \in G^{\otimes 2n} \},$$
$$H(G, e) \cap G^{\otimes 2n+1} = \mathbb{Z}\, e^{\otimes 2n+1} + e^{\otimes n} \otimes \check{G} \otimes G^{\otimes n}.$$

Hence, $(\mathbf{id} - \lambda)(z) = e \otimes z' - z' \otimes e$ and $w = ke^{\otimes 2n+1} + e^{\otimes n} \otimes x$ for some $z' \in G^{\otimes 2n}$, $k \in \mathbb{Z}$, and $x \in \check{G} \otimes G^{\otimes n}$. Therefore, by (7),

$$z' \otimes e - e \otimes z' = ke^{\otimes n+1} + e^{\otimes n} \otimes x.$$

Inductively, this equality follows that $z' = e^{\otimes n} \otimes y$ for some $y \in G^{\otimes n}$. Hence

$$y \otimes e - e \otimes y = ke^{\otimes n+1} + x.$$

By Lemma 1, we obtain $y \in \mathbb{Z}\, e^{\otimes n}$, $k = 0$, and $x = 0$. Thus, $z \in \mathbb{Z}\, e^{\otimes \mathbb{Z}}$ and $w = 0$. $\square$

Lemma 3. *Let λ be the shift on $G^{\otimes \mathbb{Z}}$. Then,*

(i) $\operatorname{Ker}(\mathbf{id} - \lambda) = \mathbb{Z}\, e^{\otimes \mathbb{Z}}$,

(ii) $G^{\otimes \mathbb{Z}} = (\mathbf{id} - \lambda)(G^{\otimes \mathbb{Z}}) \oplus H(G, e)$.

Proof. (i) At first, it is clear that $\operatorname{Ker}(\mathbf{id} - \lambda) \supset \mathbb{Z}\, e^{\otimes \mathbb{Z}}$. Conversely, take $z \in G^{\otimes \mathbb{Z}}$ such that $(\mathbf{id} - \lambda)(z) = 0$. Applying Lemma 2 to the case of $w = 0$, we obtain that $z \in \mathbb{Z}\, e^{\otimes \mathbb{Z}}$. Thus $\operatorname{Ker}(\mathbf{id} - \lambda) = \mathbb{Z}\, e^{\otimes \mathbb{Z}}$.

(ii) By Lemma 1, $(\mathbf{id} - \lambda)(G^{\otimes \mathbb{Z}}) \cap H(G, e) = 0$, so it is sufficient to show $E^{\otimes \mathbb{Z}} \subset (\mathbf{id} - \lambda)(G^{\otimes \mathbb{Z}}) \oplus H(G, e)$. Take $f = \bigotimes_{i \in \mathbb{Z}} e_i \in E^{\otimes \mathbb{Z}}$. Since $e^{\otimes \mathbb{Z}} \in H(G, e)$, we may assume $f \neq e^{\otimes \mathbb{Z}}$. Then, there exists an integer $i_0 \in \mathbb{Z}$ such that $e_{i_0} \neq e$ and $e_i = e$ for all $i < i_0$. If $i_0 = 0$, then $f \in H(G, e)$, so we may assume $i_0 \neq 0$. Note that $\lambda^{-i_0}(f) \in \check{G} \otimes G^{\otimes \mathbb{N}}$ and

$$f = \begin{cases} (\mathbf{id} - \lambda)\left(- \sum_{j=1}^{i_0} \lambda^{-j}(f) \right) + \lambda^{-i_0}(f) & \text{for } i_0 > 0 \\ (\mathbf{id} - \lambda)\left(\sum_{j=1}^{-i_0} \lambda^{-j+1}(f) \right) + \lambda^{-i_0}(f) & \text{for } i_0 < 0, \end{cases}$$

Thus, $f \in (\mathbf{id} - \lambda)(G^{\otimes \mathbb{Z}}) \oplus H(G, e)$. □

Proof of the Main Theorem. Applying Lemma 3 in the case of $G = K_*(A)$ and $e = [1_A]_0$, we obtain

$$\begin{aligned} \operatorname{Ker}(\mathbf{id} - \lambda) &= \mathbb{Z}\, [1_A]_0^{\otimes \mathbb{Z}}, \\ K_*(A)^{\otimes \mathbb{Z}} &= (\mathbf{id} - \lambda)(K_*(A)^{\otimes \mathbb{Z}}) \oplus H(K_*(A), [1_A]_0). \end{aligned}$$

Therefore, from the exactness of (2), the sequence (4) is exact. Moreover, since $\mathbb{Z}\, [1_A]_0^{\otimes \mathbb{Z}} \cong \mathbb{Z}$, (4) is split. □

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